9th CONFERENCE
on
DYNAMICAL SYSTEMS
THEORY AND APPLICATIONS
December 17-20, 2007. Łódź, Poland

STABILITY INVESTIGATION OF NONLINEAR VIBRATIONS OF PLATES BY R-FUNCTIONS METHOD

J. Awrejcewicz, L. Kurpa, and O. Mazur

Abstract: The parametric vibrations of plates with cutouts subjected to in-plane periodic and compressive loads, are studied. The proposed approach is based on R-functions method and the classical variational approach. The influence of cutouts parameters, as well as static factors of load on stability regions and nonlinear vibrations are investigated.

1. Introduction.

The most modern constructions, used in civil and aerospace engineering (and other) consist mainly various plate structures configurations. Vibration research of plates loaded by compressive pulsating force has received a particular interest recently, since in such system dynamic instability may occur yielded by certain combinations of load and eigenfrequency parameters. The so far started problem was reviewed, for instance, by Sahu and Datta [10]. On the other hand dynamic stability loss was investigated by Bolotin [2], Hutt and Salam, [4] and many others. Investigation of nonlinear parametric vibrations was carried out in [1,2,3,11], among other. However, the mentioned works are aimed on dynamic behavior analysis of regular models of rectangular plates with various boundary conditions. It is clear that in engineering practice many of components of modern civil engineering constructions have different geometry and shapes and, in particular, they may possess cutouts.

In the present work stability and nonlinear vibrations of plates with central cutouts are investigated. Both the R- function method (RFM) [5] and the variational method are applied to overcome the mathematical difficulties occurred during analysis of complex form plates. The main idea of this novel approach relies on reducing the von Kármán equations (governing dynamics of isotropic plates) to an ordinary differential equation regarding time by the Bubnov-Galerkin method. The coefficients of the obtained ordinary differential equation are found by the R-functions theory.

Instability regions and nonlinear response characteristics are studied numerically, and the influence of size of cutouts on investigated characteristics is studied.

2. Problem Formulation

The isotropic plate with constant thickness h is subjected to a uniformly distributed in-plane load $p = p_0 + p_t \cos\theta t$ along the edges $x = \pm a$, (see Figure.1), where p_0 is the static component of $p(t)$, p_t is the amplitude of the dynamic component of $p(t)$, and θ is the excitation frequency

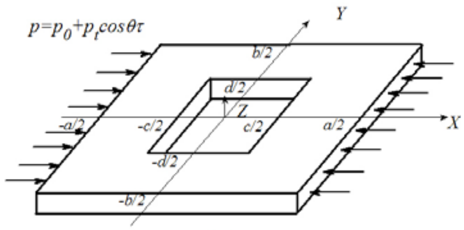

Fig.1. Form of plate

The governing equations of the nonlinear dynamics of the plate have the following form

$$\frac{\partial N_x}{\partial x} + \frac{\partial T}{\partial y} = 0, \frac{\partial N_y}{\partial y} + \frac{\partial T}{\partial x} = 0 \tag{1}$$

$$D\nabla^4 w = \left(\frac{\partial^2 w}{\partial x^2} N_x + 2\frac{\partial^2 w}{\partial x \partial y} T + \frac{\partial^2 w}{\partial y^2} N_y \right) - \rho h \frac{\partial^2 w}{\partial t^2}, \tag{2}$$

where: N_x, N_y, T are the membrane stress resultants; u, v, w are the displacements in x, y and z directions, respectively; E is the elasticity modulus; μ is the Possion's ratio; ρ is density, and $D = Eh^3/\left(12\left(1-\mu^2\right)\right)$ is the plate flexural rigidity.

Different types of the boundary conditions regarding deflection w are considered:

1) $w = 0$, $\frac{\partial w}{\partial n} = 0$ (clamped plate); (3)

2) $w = 0$, $M_n = 0$ (simply supported plate), where:

$$M_n = -D\left(\frac{\partial^2 w}{\partial n^2} + \mu \frac{\partial^2 w}{\partial \tau^2} \right); \tag{4}$$

3) mixed boundary conditions: on the outer contour $w = 0$, $\frac{\partial w}{\partial n} = 0$, and on the inner part of contour (cutout) $w = 0$, $M_n = 0$; (5)

4) on the outer contour $w = 0$, $M_n = 0$, and the inner part of contour (cutout) is free. (6)

Displacements u and v have to satisfy the following conditions

$$N_n = -p\,,\ N_\tau = 0\,,\quad x = \pm a/2\ ;$$

and we have on unloaded part of the contour: $N_n = 0$, $N_\tau = 0$. (7)

In the above n, τ are normal and tangent to domain boundary $\partial\Omega$.

The initial conditions have the following form

$$w_{/t=0} = w_0\,,\ w'_{/t=0} = 0\,.$$

Since we use further the nondimensional form of equations, relations between dimensional and nondimensional values follow

$$\bar{x} = \frac{x}{a}\,,\ \bar{y} = \frac{y}{a}\,,\ \bar{w} = \frac{w}{h}\,,\ \bar{u} = \frac{ua}{h^2}\,,\ \bar{v} = \frac{va}{h^2}\,,\ \overline{N_x} = \frac{N_x a^2}{Eh^3}\,,\ \overline{N_x} = \frac{N_x a^2}{Eh^3}\,,\ \bar{T} = \frac{Ta^2}{Eh^3}\,,$$

$$\overline{N_n} = \frac{N_n a^2}{Eh^3}\,,\ \overline{N_\tau} = \frac{N_\tau a^2}{Eh^3}\quad \bar{p} = \frac{pa^2}{Eh^3}\,,\ \bar{t} = \frac{h}{a^2}\sqrt{\frac{E}{\rho}}t\,,\ \bar{\theta} = \frac{a^2}{h}\sqrt{\frac{\rho}{E}}\theta\,,\ \overline{\omega_L} = \frac{a^2}{h}\sqrt{\frac{\rho}{E}}\omega_L\,, \tag{8}$$

where ω_L is linear frequency. In what follows the bars on nondimensional values are omitted.

Substituting expressions for N_x, N_y, T [11] into (1)-(2) and taking into account (8), equations (1)-(2) regarding displacements u, v, w can be cast into the following form :

$$A\overline{U} = \overline{NL}(w)\,, \tag{9}$$

$$\frac{1}{12(1-\mu^2)}\nabla^4 w = \frac{\partial^2 w}{\partial x^2}N_x(u,v,w) + 2\frac{\partial^2 w}{\partial x \partial y}T(u,v,w) + \frac{\partial^2 w}{\partial y^2}N_y(u,v,w) - \frac{\partial^2 w}{\partial t^2}\,, \tag{10}$$

where:

$$\overline{U} = \begin{pmatrix} u \\ v \end{pmatrix},\ A = \begin{pmatrix} \dfrac{\partial^2}{\partial x^2} + \dfrac{1-\mu}{2}\dfrac{\partial^2}{\partial y^2} & \dfrac{1+\mu}{2}\dfrac{\partial^2}{\partial x \partial y} \\ \dfrac{1+\mu}{2}\dfrac{\partial^2}{\partial x \partial y} & \dfrac{\partial^2}{\partial y^2} + \dfrac{1-\mu}{2}\dfrac{\partial^2}{\partial x^2} \end{pmatrix},$$

$$\overline{Nl}(w) = -\begin{pmatrix} \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial x^2} + \dfrac{1+\mu}{2}\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x \partial y} + \dfrac{1-\mu}{2}\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} \\ \dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial y^2} + \dfrac{1+\mu}{2}\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial x \partial y} + \dfrac{1-\mu}{2}\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x^2} \end{pmatrix}.$$

Suppose, that the studied plate is in a inhomogeneous subcriticality state, then at the first step of our method two problems of elasticity theory have to be solved, namely

$$A\overline{U}_1 = 0\,,\ A\overline{U}_2 = 0\,, \tag{11}$$

with the following boundary conditions

$N_n^L(u_1,v_1)=-p_0$, $N_\tau^L(u_1,v_1)=0$, $N_n^L(u_2,v_2)=-p_t$, $N_\tau^L(u_2,v_2)=0$, $x=\pm a/2$,

$N_n^L(u_1,v_1)=0$, $N_\tau^L(u_1,v_1)=0$, $N_n^L(u_2,v_2)=0$, $N_\tau^L(u_2,v_2)=0$,

applied to the unloaded part of counter (12), respectively, where N_n^L , N_τ^L are the normal and shear linear forces. These problems are solved using Ritz's variational method. However, a construction of a system of basic functions for the corresponding functional is carried out by the R-functions theory [5].

Hence, further investigation is reduced to linear vibration problem for the unloaded plate. In order to find the first frequency ω_L and related modal shape $w_1(x,y)$, the Ritz method and the R-functions theory are used.

The solution to nonlinear system (9)-(10) is sought for in the following form

$$w(x,y,t)=f(t)w_1(x,y),$$

$$u(x,y,t)=u_1(x,y)+u_2(x,y)\cdot cos\,\theta t+f^2(t)\cdot u_3(x,y), \tag{13}$$

$$v(x,y,t)=v_1(x,y)+v_2(x,y)\cdot cos\,\theta t+f^2(t)\cdot v_3(x,y),$$

where functions $w_1(x,y)$, $u_1(x,y)$, $v_1(x,y)$, $u_2(x,y)$, $v_2(x,y)$ are known , whereas functions $u_3(x,y)$, $v_3(x,y)$ are found by solving the inhomogeneous linear system of differential equations of the form

$$A\overline{U}_3=\overline{Nl}(w_1), \tag{14}$$

which addresses the plane elastic problem. The right hand-side of equation (14) can be considered as an action of fictitious forces, and system (14) is supplemented by the following boundary conditions

$$N_n(u_3,v_3)=0\ ,\ N_\tau(u_3,v_3)=0\ . \tag{15}$$

In order to solve the problem (14)-(15) in the case of plates with complex form, the Ritz method in combination with R-functions theory are applied.

It is easy to see that displacements u , v , w , governed by (13), satisfy the system of equations (9)-(10) with the corresponding boundary conditions.

Substituting (13) into (10) and applying the Bubnov-Galerkin method, the following nonlinear differential equation is obtained.

$$f''(t)+\omega_L^2(1-\alpha-\beta\cos\theta t)f(t)+\gamma f^3(t)=0\ , \tag{16}$$

where the coefficients α , β , γ are defined by the following relations

$$\alpha=\frac{1}{\omega_L^2\|w_1\|^2}\iint_\Omega\left(\frac{\partial^2 w_1}{\partial x^2}N_x^L(u_1,v_1)+2\frac{\partial^2 w_1}{\partial x\partial y}T^L(u_1,v_1)+\frac{\partial^2 w_1}{\partial y^2}N_y^L(u_1,v_1)\right)w_1 d\Omega\ ,$$

$$\beta = \frac{1}{\omega_L^2 \|w_1\|^2} \iint_\Omega \left(\frac{\partial^2 w_1}{\partial x^2} N_x^L(u_2, v_2) + 2\frac{\partial^2 w_1}{\partial x \partial y} T^L(u_2, v_2) + \frac{\partial^2 w_1}{\partial y^2} N_y^L(u_2, v_2) \right) w_1 d\Omega , \tag{17}$$

$$\gamma = -\frac{1}{\|w_1\|^2} \iint_\Omega \left(\frac{\partial^2 w_1}{\partial x^2} N_x(u_3, v_3, w_1) + 2\frac{\partial^2 w_1}{\partial x \partial y} T(u_3, v_3, w_1) + \frac{\partial^2 w_1}{\partial y^2} N_y(u_3, v_3, w_1) \right) w_1 d\Omega .$$

Note that in formulas (17), N_x^L, N_y^L, T^L are linear membrane stress resultants. Observe that equation (16) can be presented in the following form

$$f_t''(t) + \Omega^2 (1 - 2k \cdot \cos(\theta t)) f(t) + \gamma f^3(t) = 0 , \tag{18}$$

where $\Omega = \omega_L \sqrt{1-\alpha}$ is the vibration frequency of our plate loaded by the static component p_0, and $k = \beta/(2(1-\alpha))$ is the excitation coefficient.

3. Stability analysis

Instability regions, are determined using the Bolotin's method [2] .Note that for $\gamma = 0$, the following Mathieu equation governing linear parametric plate vibrations is obtained:

$$f_t''(t) + \Omega^2 (1 - 2k \cdot \cos(\theta t)) f(t) = 0 . \tag{19}$$

It is well known that the solutions of equations [19] can be bounded or unbounded. The boundaries between stable and unstable solutions are formed by periodic solutions with period T and $2T$, where $T = 2\pi/\theta$. Two solutions of the same period confine regions of instability (in the vicinity of $\theta = 2\Omega/r$, $r = 1,2,3,...$), whereas two solutions with different periods confine the regions of stability. Since equations of curves, governing stability and instability regions are known, the first region of instability zone is confined by the curves

$$\theta_1 = 2\Omega\sqrt{1-k} \ , \ \theta_2 = 2\Omega\sqrt{1+k} \ .$$

2. The R-functions method (RFM)

The R-functions theory is used to construct the system of basic functions for solving equations (11)-(12), (14)-(15), and the problem of linear vibrations of the unloaded plate. According to RFM, and in order to create structures of solutions, it is necessary to construct a boundary domain equation. Method of such construction is proposed in reference [5]. In the case of an arbitrary geometry, the governing equation has the form:

$$\omega(x, y) = (f_1 \wedge_0 f_2) \wedge_0 \overline{(f_3 \wedge_0 f_4)} . \tag{20}$$

The so called R-operations used in (20) are defined as follows

$$x \wedge_0 y = x + y - \sqrt{x^2 + y^2}\ ,\ x \vee_0 y = x + y + \sqrt{x^2 + y^2}\ ,\ \bar{x} = -x\ .$$

On the other hand functions f_i , i=1…5, occurred in (20), are defined my relations

$$f_1 = \frac{1}{a}\left(\left(\frac{a}{2}\right)^2 - x^2\right)\ ,\ f_2 = \frac{1}{b}\left(\left(\frac{b}{2}\right)^2 - y^2\right),\ f_3 = \frac{1}{c}\left(\left(\frac{c}{2}\right)^2 - x^2\right),\ f_4 = \frac{1}{d}\left(\left(\frac{d}{2}\right)^2 - y^2\right).$$

Note that equation of the boundary $\omega(x, y)$, constructed in such way, satisfies the following conditions

$$\omega(x, y) = 0\ ,\ \omega(x, y) > 0\ ,\ \frac{\partial\omega}{\partial n} = -1\ ,\ (x, y) \in \partial\Omega$$

Hence, the solutions structures (satisfying only principal boundary conditions) can be constructed in the following way:

1) $w_1 = \omega^2 P_0$, $u_i = P_i$, $v_i = P_{i+3}$, $i = 1..3$, for conditions (3)-(7),

2) $w_1 = \omega P_0$, $u_i = P_i$, $v_i = P_{i+3}$, $i = 1..3$, for conditions (4)-(7),

3) $w_1 = \omega_1 \omega P_0$, $u_i = P_i$, $v_i = P_{i+3}$, $i = 1..3$, for conditions (5)-(7),

4) $w_1 = \omega_1 P_0$, $u_i = P_i$, $v_i = P_{i+3}$, $i = 1..3$, for conditions (6)-(7),

where $\omega_1 = f_1 \wedge_0 f_2$ is equation governing outer part of the boundary. Uncertain components P_j , $j = 0..6$ are presented as decomposition in the series forms with a help of some complete system of functions. In this study a system of power polynomials is used.

4. Numerical results

Numerical results presented in this work are obtained only for simply supported plate with free cutout and with the applied boundary conditions (6)-(7). In order to verify our method, calculations of plate eigenfrequencies for different size of free cuts are carried out, and the obtained results are presented in the Table 1.

Table 1. Comparison of non-dimensional fundamental frequencies $\lambda = \omega_L a^2 \sqrt{\rho h / D}$ of a simply supported square plate with a cutout

c/a	Our results	[9]	[7]	[6]	[8]
0	19.742	19.734	19.739	19.740	19.739
0.2	19.708	19.134	18.901	18.762	20.193
0.4	21.108	20.739	20.556	20.785	-
0.5	23.791	23.422	23.329	23.664	24.243
0.6	28.719	28.307	28.491	28.844	-
0.8	57.948	56.949	58.847	58.062	58.359

The values of parameters used in our further analysis are: $a/b=1$, $c=d$, $\mu=0.3$. In the present work only primary region of instability is considered. The effect of the static load factor on instability zone is studied for $p_0 = 0.5, 1, 1.5$. Observe that owing to increase of static component, instability regions shift to lower frequencies. In addition, effect of cutout size is studied for different values of ratio c/a ($0 \le c/a \le 0.4$). For $p_0 = 1$, with an extension of cutout, instability region is shifted to lower frequencies up to $c/a = 0.28$. It has been observed that instability regions tend to higher frequencies from $c/a = 0.28$ to $c/a = 0.4$. For $p_0 = 2$ behavior of our system is similar to the so far described. It should be emphasized that the results for plates with cutouts are in close agreement with those obtained in reference [9].

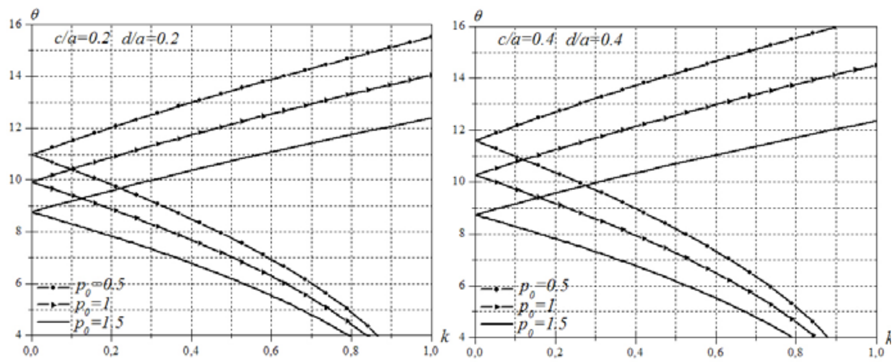

Fig.2. Effect of static load on instability region

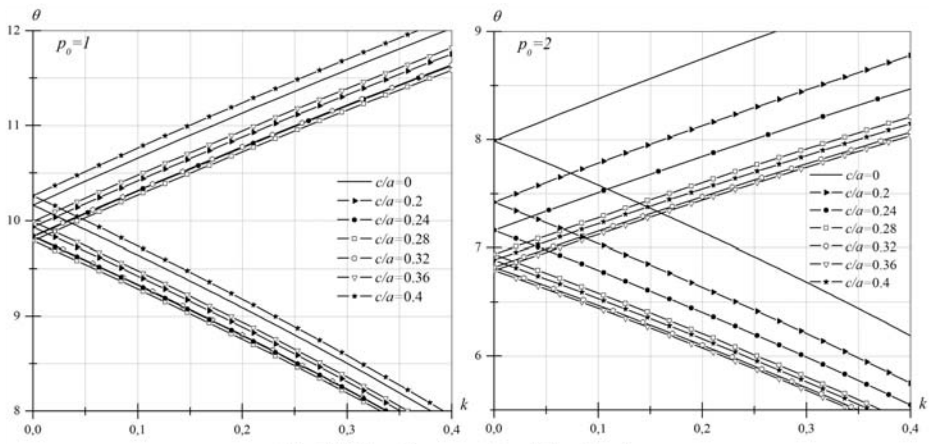

Fig. 3. Effect of cutout on instability domain

In what follows, the dynamic nonlinear response for the chosen values of load parameters $p_0 = 1$, $p_t = 1$ and with initial amplitude $w_0 / h = 0.0001$ and for $t \in [0..300]$ is studied (see Figure 4)

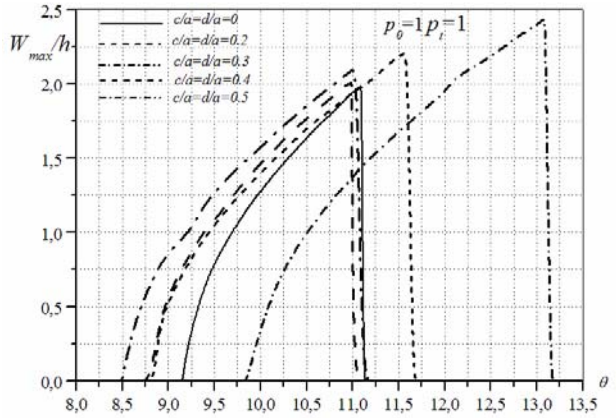

Fig.4. Non-dimensional amplitudes of plate for different size of cutouts

Observe that instability region predicated by the linear theory almost coincides with the results obtained by nonlinear analysis. It can also be seen that the amplitudes are comparable with initial conditions outside of instability region. However, for the critical zone amplitude values are increased. The values of amplitudes for various size of cutout are studied for $0 \le c / a \le 0.5$ and for $p_0 = p_t = 1$. The extension of cutouts lead to increase of the vibration amplitudes and to a movement of the response curves (for instance, a resonance zone is located between critical frequencies θ_1 and θ_2).

4. Conclusions

The obtained results can be summarized as follows: instability regions are shifted to lower frequencies with increase of static component of the load. Change of cutouts size yields to movement of instability regions i.e., first to lower frequencies, and next to higher frequencies. Nonlinear vibration analysis of plates allows to note that the extension of cutouts increases amplitudes of vibrations.

References

[1] Awrejcewicz J., Krysko A. V., *Analysis of complex parametric vibrations of plates and shells using Bubnov-Galerkin approach*. Applied Mechanics, 73, 2003, p. 495 — 504.

[2] Bolotin, V.V., *Dynamic Stability of Elastic Systems*. Gos. Tech. Izdat., Moscow, 1956, in Russian.

[3] Ganapathi M., Patel B.P., Boise P., Touratier M., Non-linear dynamic stability characteristics of elastic plates subjected to periodic in-plane load. *International Journal of Non-linear Machanics*, 35, 2000, p. 467-480.

[4] Hutt J.M., Salam A.E., B.P., Dynamic stability of plates by finite element. *Journal of Engineerinrg Mechanics*, 97, 1971, p. 879-899.

[5] Rvachev V.L., Kurpa L.V., *R-functions in problems of theory of plates*. Naukova Dumka, Kiev, 1987, in Russian.

[6] Lam K.Y., Hung K.C., Orthogonal polynomials and subsectioning method for vibration of plates. *Computers and Structures*, 34, 1990, p.827-834.

[7] Lee H.P., Lim S.P., Chow S.T., Prediction of natural frequencies of rectangular plates with rectangular cutouts. *Computers and Structures*, 36, 1990, p.861-869.

[8] Mundkur G., Bhat R.B. and Neriya S., Vibration of plates with cutouts using boundary characteristics orthogonal polynomial functions in the Rayleigh-Ritz method. *Journal of Sound and Vibration*, 176, 1994, p.136-144.

[9] Sahu S.K., Datta P.K., Dynamic stability of curved panels with cutouts. *Journal of Sound and Vibration*, 251(4), 2002, p.683-696.

[10] Sahu S.K., Datta P.K., Research Advances in the dynamic stability behavior of plates and shells: 1987-2005 — Part1: conservative system. *Applied Mechanics Reviews*, 60, 2007, p.65-75.

[11] Volmir, A. S., *Nonlinear dynamics of plates and shells*. Nauka, Moscow, 1972, in Russian.

Jan Awrejcewicz
Department of Automatics and Biomechanics, Technical University of Lodz
E-mail: awrejcew@p.lodz.pl

Lidiya V. Kurpa
Department of Applied Mathematics, NTU "Kharkov Politechnical Institute"
21 Frunze st, 61002,. Kharkov, Ukraine
E-mail: Kurpa@kpi.kharkov.ua

Olga S. Mazur
Department of Applied Mathematics, NTU "Kharkov Politechnical Institute"
21 Frunze str, 61002,. Kharkov, Ukraine
E-mail: OlgaMazur@ukr.net